THREE TYPES OF TASKS
- EXTENDED INVESTIGATIONS
- IN-CLASS INVESTIGATIONS
- INVESTIGATIVE QUESTIONS

18 ASSESSMENT TASKS

INCLUDES SOLUTIONS WITH MARKING KEYS SHOWING MATHEMATICAL BEHAVIOURS

WRITTEN BY TEACHERS FOR TEACHERS

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Australian CURRICULUM

The Mathematical Association of Western Australia Inc.
MAΣWA

FREE SAMPLE TASK

INVESTIGATIONS FOR TEACHING & LEARNING YEAR 12 SPECIALIST MATHEMATICS

Foreword

For each unit three types of tasks are included:

- **Extended teaching and learning investigation** – an investigation that includes a preparation activity which could be done in class, individually or in groups, in the student's own time **followed by** an in-class validation, so that students can reflect on their understanding
- Solutions are provided where practicable for the preparation activity/investigation. For the in-class validation, there are solutions and marking keys, which identify the mathematical behaviours that students may exhibit.
- Teachers are encouraged to share these with students, so they gain experience with WACE style marking keys.

- **In-class teaching and learning investigation** – an investigation for which no prior preparation is required.
- Solutions and marking keys are provided.
- This type of task will give students experience with investigative style questions that may be included as part of response type assessments.

- **Investigative questions** – a series of short questions, which test the student's ability to apply their learning, to justify their conclusions, to investigate and to generalise, or to solve problems.
- Such questions could be included in a response or examination assessment. Solutions and marking key are provided.

Screen shots have been produced using the CASIO ClassPad II emulator

Most graphs and diagrams have been produced using efofex software.

CONTENTS

SPECIALIST MATHEMATICS Year 12

TASK 11

SURFACE AREA

In-class investigation

Unit 4

Topic 4.1: Integration and applications of integration

Course-related information

The concepts and skills included in this investigation relate to the following dot points within the ACARA Specialist Mathematics syllabus:

- use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))g'(x)$ (ACMSM117)
- determine volumes of solids of revolution about either axis (ACMSM125).

Background information

The mathematical skills and understandings that students would need for this task should have been covered in earlier years and in Mathematical Methods Unit 3: Topic 2. It is also assumed that students have already covered Specialist Mathematics ACMSM117 and ACMSM125 in class.

Task conditions

For this task students should not have access to a CAS calculator. No student notes are required; formulae are provided.

SURFACE AREA

In-class investigation (Total marks: 40)

When a plane curve is rotated about an axis, a hollow three-dimensional solid is created. The surface area of this solid may be determined.

The point (x, y) on the curve traces out a circle whose circumference is $2\pi y$ or $2\pi x$ as it is rotated about the x-axis or y-axis .

The area of a surface of revolution generated by revolving the curve $y = f(x),\ a \le x \le b,$ about the $x-$axis is given by

$$S_x = \int_{x=a}^{x=b} 2\pi y\, ds$$

The area of a surface of revolution generated by revolving the curve $x = g(y),\ c \le y \le d,$ about the $y-$axis is given by

$$S_y = \int_{y=c}^{y=d} 2\pi x\, ds$$

where $(ds)^2 = (dx)^2 + (dy)^2$.

Question 1 (13 marks)

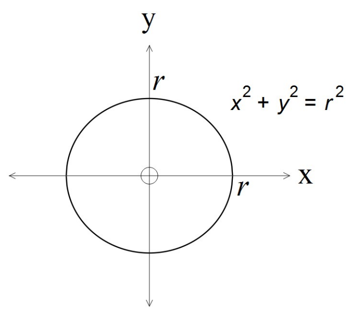
Determine the area of the surface of revolution obtained by rotating the arc of the parabola $y = x^2$, $1 \le x \le 3$, about the $y -$axis using:

(a) $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ (6)

(b) $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ (5)

(c) Comment on your results. (2)

Question 2 (7 marks)



Show that the surface area of a sphere of radius r is $4\pi r^2$ by rotating an arc of the curve $x^2 + y^2 = r^2$ about the $x-$axis.

Question 3 (12 marks)

The formula for the volume and surface area of a cone of height h and radius r may be determined by rotating the line segment $y = mx,\ 0 \le x \le a,$ about the $x-$axis.

(a) Express m and a in terms of h and/or r. (2)

Hence, by rotating this line segment about the $x-$axis,

(b) show that the volume of the cone is given by $V_{cone} = \frac{1}{3}\pi r^2 h$. (5)

(c) determine a formula for the surface area of the cone. (5)

Question 4 (5 marks)

A group of Year 12 students is building a satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the $y-$axis. If the dish is to have a diameter of 3 m and a maximum depth of 0.6 m, determine the exact surface area of the dish.

Question 5 (3 marks)

If the curve $y = f(x)$, $a \le x \le b$, is rotated about the horizontal line $y = c$, where $f(x) \le c$, determine a formula for the area of the resulting surface.

SURFACE AREA

In-class investigation
Solutions and marking key

Question 1(a)

Solution
Using $S_y=\int_{y=c}^{y=d} 2\pi x\,ds$ where $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\ c=1,\ d=9$ $y=x^2$ $\frac{dy}{dx}=2x$ $S_y=2\pi\int\limits_{y=1}^{y=9} x\sqrt{1+(2x)^2}\ dx$ $=2\pi\int\limits_{x=1}^{x=3} x\sqrt{1+4x^2}\,dx,$ let $u=1+4x^2,\ du=8x\,dx,$ $x=1,\ u=5,$ $x=3,\ u=37$ $=\frac{\pi}{4}\int\limits_{5}^{37}\sqrt{u}\,du$ $=\frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^{37}$ $=\frac{\pi}{6}\left(\sqrt{37^3}-\sqrt{5^3}\right)$ $=\frac{\pi}{6}\left(37\sqrt{37}-5\sqrt{5}\right)$

Mathematical behaviours	Marks
• Chooses $S_y=\int_{y=c}^{y=d} 2\pi x\,ds$	1
• Correct upper and lower limits	1
• Substitutes correctly for $\frac{dy}{dx}$	1
• Recognises integral of the form $\left(f\left(g(x)\right)\right)g'(x)$	1
• Correct antiderivative	1
• Correct surface area	1

Question 1(b)

Solution
Using $S_y = \int_{y=c}^{y=d} 2\pi x \, ds$ where $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,\ c=1,\ d=9$ $y = x^2,\ 1 \le x \le 3 \quad \Rightarrow \quad x = \sqrt{y}$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}},\ 1 \le y \le 9$ $S_y = 2\pi \int_1^9 \sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\ dy$ $= 2\pi \int_1^9 \sqrt{y}\sqrt{1+\frac{1}{4y}}dy$ $= 2\pi \int_1^9 \sqrt{y+\frac{1}{4}}dy$ $= 2\pi \left[\frac{2}{3}\left(y+\frac{1}{4}\right)^{\frac{3}{2}}\right]_1^9$ $= \frac{4\pi}{3}\left(\sqrt{\left(\frac{37}{4}\right)^3} - \sqrt{\left(\frac{5}{4}\right)^3}\right)$ $= \frac{\pi}{6}\left(37\sqrt{37} - 5\sqrt{5}\right)$

Mathematical behaviours	Marks
• Correct upper and lower limits	1
• Substitutes correctly for $\frac{dx}{dy}$	1
• Simplifies in terms of y	1
• Correct antiderivative	1
• Correct surface area	1

Question 1(c)

Solution
Concludes that the answers to (a) and (b) are the same because $$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(dx)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)} \\ &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \end{aligned}$$ and $$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(dy)^2\left(\left(\frac{dx}{dy}\right)^2+1\right)} \\ &= \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \end{aligned}$$

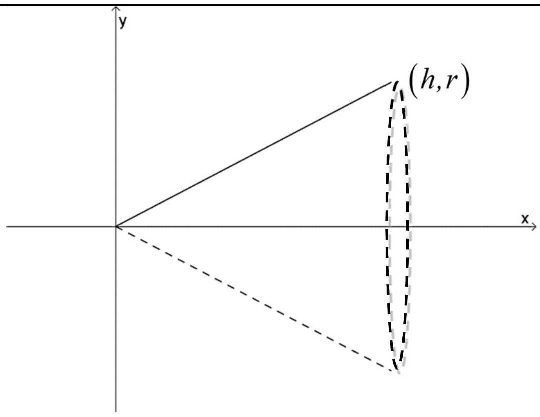
Mathematical behaviours	Marks
• Recognises results are the same • Supports conclusion with reasoning	1 1

Question 2

Solution
Rotate $y=\sqrt{r^2-x^2}$, $-r\le x\le r$, about the $x-$axis.
Using $S_x=\int_{x=a}^{x=b} 2\pi y\, ds$,
where $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ $=\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$ $=\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx$ $=\frac{r}{\sqrt{r^2-x^2}}\,dx$
Hence, $S_x=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}.\frac{r}{\sqrt{r^2-x^2}}\,dx$ $=2\pi r\int_{-r}^{r} dx$ $=2\pi r\left[x\right]_{-r}^{r}$ $=2\pi r\left[2r\right]$ $=4\pi r^2$
Hence, surface area of a sphere of radius r is $4\pi r^2$.

Mathematical behaviours	Marks
• rotates an arc of the circle about the $x-$axis	1
• uses correct integral for area of surface of revolution about $x-$axis	1
• correct expression for $\frac{dy}{dx}$	1
• simplifies expression for ds	1
• simplifies integral expression	1
• correct antiderivative	1
• correct conclusion	1

Question 3(a)

Solution

$m=\dfrac{r}{h}, \qquad a=h$

Mathematical behaviours	Marks
• Correct expression for m • Correct expression for a	1 1

Question 3(b)

Solution
Rotate the line segment $y=\dfrac{r}{h}x,\ 0\le x\le h$, about the $x-$axis,
$\begin{aligned} V_{cone} &= \pi\int_0^h y^2\,dx \\ &= \pi\int_0^h\left(\frac{r}{h}x\right)^2 dx \\ &= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$

Mathematical behaviours	Marks
• Chooses correct integral • Correct upper and lower limits • Substitutes for y in terms of x • Correct antiderivative • Establishes formula for volume of a cone	1 1 1 1 1

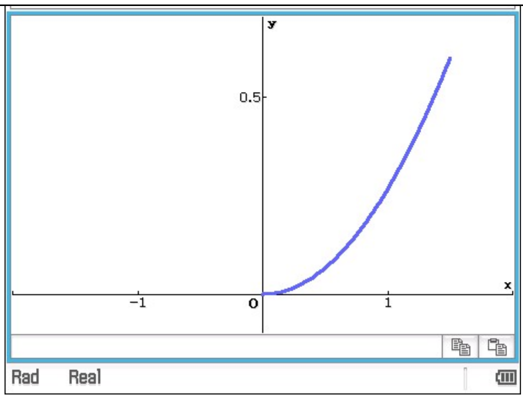
Question 3(c)

Solution
Rotate the line segment $y = \frac{r}{h}x,\ 0 \le x \le h$, about the $x-$axis, $S_x = \int_{x=a}^{x=b} 2\pi\, y\, ds$ $S_{cone} = 2\pi \int_0^h y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $= 2\pi \int_0^h \frac{r}{h}x \sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx$ $= 2\pi\, \frac{r}{h^2} \sqrt{r^2 + h^2} \int_0^h x\, dx$ $= 2\pi\, \frac{r}{h^2} \sqrt{r^2 + h^2} \left[\frac{x^2}{2}\right]_0^h$ $= 2\pi\, \frac{r}{h^2} \sqrt{r^2 + h^2} . \frac{h^2}{2}$ $= \pi r \sqrt{r^2 + h^2}$

Mathematical behaviours	Marks
• Chooses correct integral	1
• Correct upper and lower limits	1
• Substitutes for y in terms of x and for $\frac{dy}{dx}$	1
• Correct antiderivative	1
• Establishes formula for volume of a cone	1

Question 4

Solution



Curve passes through $(1.5, 0.6)$,

$$0.6 = a(1.5)^2$$

$$\Rightarrow \quad a = \frac{4}{15}$$

$$y = \frac{4}{15}x^2, \quad 0 \le x \le 1.5$$

$$S_y = \int_0^{1.5} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^{1.5} x \sqrt{1 + \left(\frac{8}{15}x\right)^2}\, dx$$

$$= 2\pi \int_0^{1.5} x \sqrt{1 + \frac{64}{225}x^2}\, dx$$

Let $u = 1 + \frac{64}{225}x^2$, $du = \frac{128}{225}x\, dx$

$x = 0,\ u = 1 \qquad x = 1.5,\ u = \frac{16}{25}$

$$= \frac{225\pi}{64} \int_1^{\frac{16}{25}} \sqrt{u}\, du$$

$$= \frac{225\pi}{64} \left[\frac{1}{2\sqrt{u}}\right]_1^{\frac{16}{25}}$$

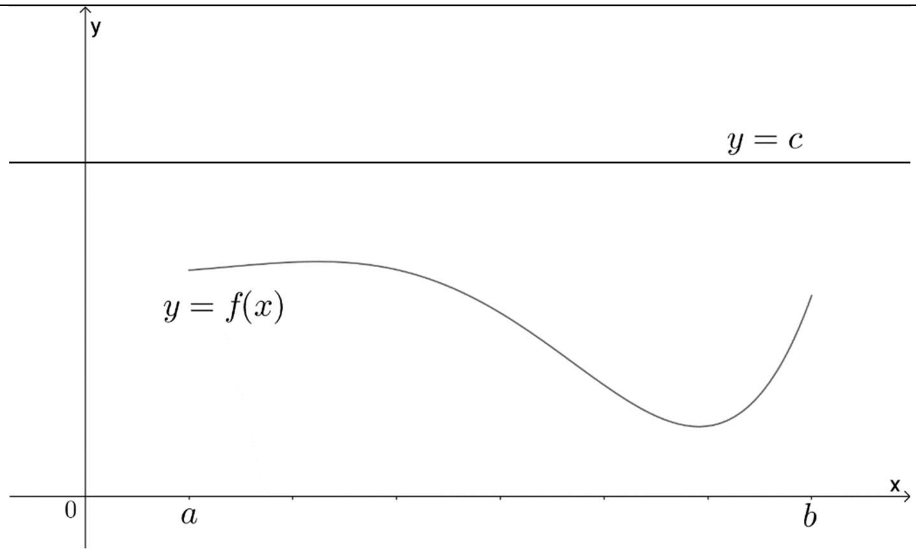
$$= \frac{225\pi}{128} \left[\frac{5}{4} - 1\right]$$

$$= \frac{225\pi}{512} \text{ m}^2$$

Mathematical behaviours	Marks
• Determines a	1
• Chooses correct definite integral	1
• Substitutes for $\frac{dy}{dx}$	1
• Correct antiderivative	1
• Calculates surface area	1

Question 5

Solution



Let (x, y) be a point on the curve $y = f(x)$.

The radius of the circle formed when this point is rotated about the line $y = c$ is $(c - y)$.

The circumference of the circle formed when this point is rotated about the line $y = c$ is $2\pi(c - y)$.

$$S_{y=c} = \int_a^b 2\pi(c - y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_a^b \left(c - f(x)\right)\sqrt{1 + \left(f'(x)\right)^2}\,dx$$

Mathematical behaviours	Marks
• Establishes radius of circle	1
• Establishes circumference of circle	1
• Determines formula for surface area	1